CMPT 476/981: Introduction to Quantum Algorithms
Assignment 1

Due **January 18, 2024 at 11:59pm on coursys**
Complete individually and submit in PDF format.

Question 1 [3 points]: A universal classical gate

The $NAND$ gate is a classical gate with the following truth table:

x	y	$NAND(x, y)$
0	0	1
0	1	1
1	0	1
1	1	0

1. Show that the NOT gate can be implemented with NAND gates and FANOUT. You may draw a circuit or simply give the algebraic expression.
2. Show that the gate set $\{NAND, FANOUT\}$ is universal for classical computation by giving implementations of each gate in the universal gate set $\{AND, OR, NOT, FANOUT\}$.

Question 2 [6 points]: Dirac notation

Let $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{i}{\sqrt{3}}|1\rangle + \frac{-1}{\sqrt{3}}|2\rangle$, $|\phi\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{-i}{\sqrt{2}}|2\rangle$ be two states of a **qutrit** (i.e. a three-level or three-dimensional system).

1. Give the explicit column vectors of $|\psi\rangle$ and $|\phi\rangle$
2. Calculate the following:
   - $\langle\psi|\psi\rangle$
   - $\langle\phi|\phi\rangle$
   - $\langle\psi|\phi\rangle$
   - $|\psi\rangle\langle\phi|$
   - $|\psi\rangle \otimes |\phi\rangle$
3. Is the vector $|\psi\rangle + |\phi\rangle$ a unit vector? If not, normalize it to get a unit vector.

Question 3 [4 points]: Gates and measurement

Suppose we have a qubit initially in the state $\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$ for some $\theta \in \mathbb{R}$.

1. Calculate the probabilities of receiving result "0" or "1" if the qubit is measured.
2. Recall the definition of the Hadamard gate, which has the vectors $|+\rangle$ and $|-\rangle$ as its columns:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   If we first apply the Hadamard gate to the initial state $\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$ and then measure, what are the probabilities of receiving the "0" and "1" results as an function of θ?
   Note: this is the same thing as measuring the initial state in the $|+\rangle, |-\rangle$ basis.

Question 4 [5 points]: Eigenvectors

Recall that an *eigenvector* of a matrix A is a vector $|v\rangle$ such that $A|v\rangle = \lambda|v\rangle$ for some scalar *eigenvalue* λ.

1. Let $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. Find two **unit** vectors $|+_Y\rangle$, $|-_Y\rangle$ such that

$$Y|+_Y\rangle = |+_Y\rangle$$
$$Y|-_Y\rangle = -|-_Y\rangle$$

2. Let U be the 2 by 2 matrix with columns $|+_Y\rangle$ and $|-_Y\rangle$. Is U unitary?
3. Calculate $U^\dagger Y U$.